

GROUPS THAT HAVE THE SAME HOLOMORPH OF A FINITE PERFECT GROUP

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ABSTRACT. We describe the groups that have the same holomorph of a finite perfect group. Our results are complete for centerless groups.

When the center is non-trivial, some questions remain open. The peculiarities of the general case are illustrated by a couple of examples that might be of independent interest.

1. INTRODUCTION

We are concerned with the question, when do two groups have the same holomorph? Recall that the *holomorph* of a group G is the natural semidirect product $\text{Aut}(G)G$ of G by its automorphism group $\text{Aut}(G)$. To put this problem in proper context, recall that if $\rho : G \rightarrow S(G)$ is the right regular representation of G , where $S(G)$ is the group of permutations on the set G , then $N_{S(G)}(\rho(G)) = \text{Aut}(G)\rho(G)$ is isomorphic to the holomorph of G . We will also refer to $N_{S(G)}(\rho(G))$ as the holomorph of G , and write the latter as $\text{Hol}(G)$. More generally, if $N \leq S(G)$ is a regular subgroup, then $N_{S(G)}(N)$ is isomorphic to the holomorph of N . We therefore begin to make the above question more precise by asking for which regular subgroup N of $S(G)$ one has $N_{S(G)}(N) = \text{Hol}(G)$.

W.H. Mills has noted [Mil51] that such an N need not be isomorphic to G (see Example 3.1, but also Example 1.2 below, and the comment following it). In this paper, we will take the following

Definition 1.1. We will say that N *has the same holomorph as G* if N is an element of the set

$$\mathcal{H}(G) = \left\{ N \leq S(G) : N \text{ is regular, } N \cong G \text{ and } N_{S(G)}(N) = \text{Hol}(G) \right\}.$$

G.A. Miller has shown [Mil08] that the the so-called *multiple holomorph* of G

$$\text{NHol}(G) = N_{S(G)}(\text{Hol}(G))$$

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acts transitively on $\mathcal{H}(G)$, and thus the group

$$T(G) = \text{NHol}(G) / \text{Hol}(G)$$

acts regularly on $\mathcal{H}(G)$.

Recently T. Kohl has described [Koh15] the set $\mathcal{H}(G)$ and the group $T(G)$ for G dihedral or generalized quaternion. In [CDV16], we have redone, via a commutative ring connection, the work of Mills [Mil51], which determined $\mathcal{H}(G)$ and $T(G)$ for G a finitely generated abelian group.

In this paper we consider the case when G is a finite, *perfect* group, that is, G equals its derived subgroup G' .

If G has also trivial center, then one can show that if $N \trianglelefteq \text{Hol}(G)$ is a regular subgroup, then $N \in \mathcal{H}(G)$ (in particular, $N \cong G$). The elements of $\mathcal{H}(G)$ can be described in terms of a Krull-Remak-Schmidt decomposition of G as a group with $\text{Aut}(G)$ as group of operators, that is, in terms of the unique decomposition of G as a direct product of non-trivial characteristic subgroups that are indecomposable as the direct product of characteristic subgroups. The group $T(G)$ turns out to be an elementary abelian 2-group.

If G has non-trivial center, the regular subgroups N such that $N \trianglelefteq \text{Hol}(G)$ can still be described in terms of the decomposition of G as the central product of non-trivial, perfect, characteristic subgroups, that are indecomposable as a central product of characteristic subgroups.

However, these N need not be isomorphic to G (see Example 1.2 below, and the comment following it), and the structure of $T(G)$ in this case is not clear to us at the moment. The difficulties here are illustrated by the following examples, which might be of independent interest.

Example 1.2. There is a group G which is the central product of two characteristic subgroups L_1, L_2 , such that G is not isomorphic to the group (G, \circ) obtained from G by replacing L_1 with its opposite.

Recall the the opposite of a group L is the group obtained by exchanging the order of factors in the product of L .

We will see in Section 7 that the group (G, \circ) is isomorphic to a regular subgroup N of $S(G)$ such that $N_{S(G)}(N) = \text{Hol}(G)$. Therefore in our context the latter condition does not imply $N \cong G$.

Example 1.3. There is a group G which is the central product of three characteristic subgroups L_1, L_2, L_3 such that L_1 and L_2 are not characteristic in the group obtained from G by replacing L_1 with its opposite.

As we will see in Section 7, this example shows that if $N \trianglelefteq \text{Hol}(G)$ is a regular subgroup, for G perfect, we may well have that $N_{S(G)}(N)$ properly contains $\text{Hol}(G)$.

Example 1.2 and 1.3 are given in Subsection 7.2 as Proposition 7.10 and Proposition 7.9.

The plan of the paper is the following. Sections 2 and 3 introduce the holomorph and the multiple holomorph. In these sections, and in the following ones, we have chosen to repeat some elementary and well-known arguments, when we have deemed them handy for later usage. In Sections 4 and 5 we give a description of the regular subgroups N of $\text{Hol}(G)$, and of those that are normal in $\text{Hol}(G)$, in terms of a certain map $\gamma : G \rightarrow \text{Aut}(G)$. This leads to a group operation \circ on G such that N is isomorphic to (G, \circ) . In Section 6 we show that the values of γ on commutators are inner automorphisms, and this leads us to consider perfect groups.

In Section 7 we study the case of finite, perfect groups. We first obtain a description of the normal subgroups of $\text{Hol}(G)$ that are regular, in terms of certain central product decompositions of G (Theorem 7.5). We then discuss separately, as explained above, the centerless case, where we can give a full picture, and the general case, where some questions remain open.

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2. THE HOLOMORPH OF A GROUP

The holomorph of a group G is the natural semidirect product

$$\text{Aut}(G)G$$

of G by its automorphism group $\text{Aut}(G)$. Let $S(G)$ be the group of permutations on the set G . Consider the right and the left regular representations of G :

$$\left\{ \begin{array}{l} \rho : G \rightarrow S(G) \\ g \mapsto (x \mapsto xg) \end{array} \right\} \quad \left\{ \begin{array}{l} \lambda : G \rightarrow S(G) \\ g \mapsto (x \mapsto gx) \end{array} \right\}.$$

Notation 2.1. We denote the inversion map $g \mapsto g^{-1}$ on a group by inv .

The following is well-known.

Proposition 2.2.

- (1) $C_{S(G)}(\rho(G)) = \lambda(G)$ and $C_{S(G)}(\lambda(G)) = \rho(G)$.
- (2) Inversion on G normalizes $\text{Hol}(G)$, centralizes $\text{Aut}(G)$, and conjugates $\rho(G)$ to $\lambda(G)$, that is

$$\rho(G)^{\text{inv}} = \lambda(G).$$

- (3) $N_{S(G)}(\rho(G)) = \text{Aut}(G)\rho(G) = \text{Aut}(G)\lambda(G) = N_{S(G)}(\lambda(G))$ is isomorphic to the holomorph $\text{Aut}(G)G$ of G .

Notation 2.3. We write $\text{Hol}(G) = N_{S(G)}(\rho(G))$.

We will refer to either of the isomorphic groups $N_{S(G)}(\rho(G))$ and $\text{Aut}(G)G$ as the holomorph of G .

Notation 2.4. We write permutations as exponents, and denote compositions of maps by juxtaposition.

We now record another well-known fact.

Lemma 2.5.

- (1) *Let Ω be a set, and G a regular subgroup of $S(\Omega)$. Then there is an isomorphism $S(\Omega) \rightarrow S(G)$ that sends G to $\rho(G)$, and thus $N_{S(\Omega)}(G)$ to $\text{Hol}(G)$.*
- (2) *If $N \leq S(G)$ is a regular subgroup, then $N_{S(G)}(N)$ is isomorphic to the holomorph of N .*

It is because of Lemma 2.5(1) that we have done without a set Ω , and started directly with $S(G)$ and its regular subgroup $\rho(G)$.

Proof. We only treat (2), for further reference.

Consider for such a regular subgroup N the bijection

$$\begin{aligned} \varphi : N &\rightarrow G \\ n &\mapsto 1^n. \end{aligned}$$

Then

$$\begin{aligned} \psi : S(G) &\rightarrow S(N) \\ \sigma &\mapsto \varphi \sigma \varphi^{-1} \end{aligned}$$

(where we compose left-to-right) is an isomorphism, which maps N onto $\rho(N)$, as for $x, n \in N$ we have

$$x^{\varphi n \varphi^{-1}} = (1^{xn})^{\varphi^{-1}} = xn,$$

that is,

$$\varphi n \varphi^{-1} = \rho(n).$$

In particular, $\psi(N_{S(G)}(N)) = N_{S(N)}(\rho(N)) = \text{Hol}(N)$. □

3. GROUPS WITH THE SAME HOLOMORPH

In view of Lemma 2.5(2), one may inquire, what are the regular subgroups $N \leq S(G)$ for which

$$(3.1) \quad \text{Hol}(N) \cong N_{S(G)}(N) = N_{S(G)}(\rho(G)) = \text{Hol}(G).$$

W.H. Mills has noted in [Mil51] that if (3.1) holds, then G and N need not be isomorphic.

Example 3.1. One can verify, for instance using GAP [GAP16], that the dihedral group G and the dicyclic group N of order 12 in suitable regular representations have the same normalizer in S_{12} .

When we restrict our attention to the regular subgroups N of $S(G)$ for which $N_{S(G)}(N) = \text{Hol}(G)$ and $N \cong G$, we can appeal to a result of G.A. Miller [Mil08]. Miller found a characterization of these subgroups in terms of the *multiple holomorph* of G

$$\text{NHol}(G) = N_{S(G)}(\text{Hol}(G)).$$

Consider the set

$$\mathcal{H}(G) = \{N \leq S(G) : N \text{ is regular, } N \cong G \text{ and } N_{S(G)}(N) = \text{Hol}(G)\}.$$

According to Definition 1.1, we will say that the element of $\mathcal{H}(G)$ are the *groups that have the same holomorph as G* .

Using the well-known fact that two regular subgroups of $S(G)$ are isomorphic if and only if they are conjugate in $S(G)$, Miller showed that the group $\text{NHol}(G)$ acts transitively on $\mathcal{H}(G)$ by conjugation. (See Lemma 4.2 in the next Section for a comment on this.) Clearly the stabilizer in $\text{NHol}(G)$ of any element $N \in \mathcal{H}(G)$ is $N_{S(G)}(N) = \text{Hol}(G)$. We obtain

Theorem 3.2. *The group*

$$T(G) = \text{NHol}(G) / \text{Hol}(G)$$

acts regularly on $\mathcal{H}(G)$ by conjugation.

4. REGULAR SUBGROUPS OF THE HOLOMORPH

This sections adapts to the nonabelian case the results of [CDVS06, Theorem 1] and [FCC12, Proposition 2].

Let G be a finite group, and $N \leq S(G)$ a regular subgroup. Write $\nu(g)$, with $g \in G$, for the unique element of N such that $1^{\nu(g)} = g$. Then for $g \in G$ we can write uniquely

$$(4.1) \quad \nu(g) = \gamma(g)\rho(g),$$

for a suitable map $\gamma : G \rightarrow \text{Aut}(G)$. We have

$$(4.2) \quad \nu(g)\nu(h) = \gamma(g)\rho(g)\gamma(h)\rho(h) = \gamma(g)\gamma(h)\rho(g^{\gamma(h)}h).$$

Since N is a subgroup of $S(G)$, and the expression (4.1) is unique, we obtain

$$(4.3) \quad \gamma(g)\gamma(h) = \gamma(g^{\gamma(h)}h).$$

It is now immediate to obtain

Theorem 4.1. *Let G be a finite group. The following data are equivalent.*

- (1) *A regular subgroup $N \leq S(G)$.*
- (2) *A map $\gamma : G \rightarrow \text{Aut}(G)$ such that*

$$(4.4) \quad \gamma(g)\gamma(h) = \gamma(g^{\gamma(h)}h).$$

Moreover, under these assumptions

(1) *the assignment*

$$g \circ h = g^{\gamma(h)} h.$$

for $g, h \in G$, defines a group structure (G, \circ) with the same unity as that of G .

(2) *There is an isomorphism $\nu : (G, \circ) \rightarrow N$.*

(3) *For $g, h \in G$, one has*

$$g^{\nu(h)} = g \circ h.$$

Proof. Concerning the last statements, (4.4) implies that \circ is associative. Then for each $h \in G$ one has that $1 \circ h = 1^{\gamma(h)} h = h$, as $\gamma(h) \in \text{Aut}(G)$, and that $(h^{-1})^{\gamma(h)^{-1}}$ is a left inverse of h with respect to \circ . The bijection ν introduced above is a homomorphism $(G, \circ) \rightarrow N$ by (4.2) and (4.3). Finally,

$$g^{\nu(h)} = g^{\gamma(h)} h = g \circ h.$$

□

Note, for later usage, that (4.4) can be rephrased, setting $k = g^{\gamma(h)}$, as

$$(4.5) \quad \gamma(kh) = \gamma(k^{\gamma(h)^{-1}}) \gamma(h).$$

We record the following Lemma, which will be useful later. We use the setup of Theorem 4.1.

Lemma 4.2. *Suppose $N \in \mathcal{H}(G)$, and let $\vartheta \in \text{NHol}(G)$ such that $\rho(G)^{\vartheta} = N$ and $1^{\vartheta} = 1$. Then*

$$\vartheta : G \rightarrow (G, \circ)$$

is an isomorphism.

Conversely, an isomorphism $\vartheta : G \rightarrow (G, \circ)$ conjugates $\rho(G)$ to N .

Proof. Note first that given any $\vartheta \in \text{NHol}(G)$ such that $\rho(G)^{\vartheta} = N$, we can modify ϑ by a suitable $\rho(g)$, and assume $1^{\vartheta} = 1$.

Suppose for $y \in G$ one has $\rho(y)^{\vartheta} = \nu(y^{\sigma})$, for some $\sigma \in S(G)$. Thus $\rho(y)\vartheta = \vartheta\nu(y^{\sigma})$, so that for $x, y \in G$ one has

$$(xy)^{\vartheta} = x^{\rho(y)\vartheta} = x^{\vartheta\nu(y^{\sigma})} = x^{\vartheta\gamma(y^{\sigma})} y^{\sigma} = x^{\vartheta} \circ y^{\sigma}.$$

Setting $x = 1$ we see that $\vartheta = \sigma$, and thus

$$(xy)^{\vartheta} = x^{\vartheta} \circ y^{\vartheta}.$$

For the converse, if the last equation holds then

$$x^{\rho(y)^{\vartheta}} = (x^{\vartheta^{-1}} y)^{\vartheta} = x \circ y^{\vartheta} = x^{\nu(y^{\vartheta})}.$$

□

5. NORMAL REGULAR SUBGROUPS OF THE HOLOMORPH

In this section, we adapt to the nonabelian case the results of [CDV16, Theorem 3.1].

Consider the sets

$$\mathcal{K}(G) = \{N \leq S(G) : N \text{ is regular, } N \trianglelefteq \text{Hol}(G)\}$$

and

$$\mathcal{L}(G) = \{N \leq S(G) : N \text{ is regular, } N_{S(G)}(N) = \text{Hol}(G)\}.$$

Clearly if $N \trianglelefteq \text{Hol}(G)$, then $\text{Hol}(G) \leq N_{S(G)}(N)$, so that we have

$$(5.1) \quad \mathcal{H}(G) \subseteq \mathcal{L}(G) \subseteq \mathcal{K}(G).$$

However, if $\text{Hol}(G) \leq N_{S(G)}(N)$, the latter may well be properly bigger than the former, as shown by the following simple example.

Example 5.1. Let $G = \langle (1\,2\,3\,4) \rangle \leq S_4$. Then $N_{S_4}(G)$ has order 8, but its regular subgroup $N = \langle (1\,3)(2\,4), (1\,4)(2\,3) \rangle$ is normal in the whole S_4 .

Moreover, even when $\text{Hol}(G) = N_{S(G)}(N)$, Example 3.1 shows that G and N are not necessarily isomorphic. Therefore inclusions in (5.1) may well be proper.

We will now give a characterization of the elements of $\mathcal{K}(G)$ in terms of the description of Theorem 4.1. Suppose $N \in \mathcal{K}(G)$. To ensure that $N \trianglelefteq \text{Hol}(G)$, it is enough to make sure that N is normalized by $\text{Aut}(G)$. In fact, if this holds, then the normalizer of N contains $\text{Aut}(G)N$, which is contained in $\text{Hol}(G)$, and has the same order as $\text{Hol}(G)$, as the regular subgroup N intersects $\text{Aut}(G)$ trivially.

In order for $\text{Aut}(G)$ to normalize N , we must have that for all $\beta \in \text{Aut}(G)$ and $g \in G$, the conjugate $\nu(g)^\beta$ of $\nu(g)$ by β in $S(G)$ lies in N . Since

$$\nu(g)^\beta = (\gamma(g)\rho(g))^\beta = \gamma(g)^\beta \rho(g)^\beta = \gamma(g)^\beta \rho(g^\beta),$$

uniqueness of (4.1) implies

$$(5.2) \quad \gamma(g^\beta) = \gamma(g)^\beta$$

for $g \in G$ and $\beta \in \text{Aut}(G)$. Applying this to (4.5), we obtain

$$(5.3) \quad \gamma(kh) = \gamma(k^{\gamma(h)^{-1}})\gamma(h) = \gamma(k)^{\gamma(h)^{-1}}\gamma(h) = \gamma(h)\gamma(k),$$

that is, $\gamma : G \rightarrow \text{Aut}(G)$ is an antihomomorphism.

Now note that (4.4) follows from (5.2) and (5.3), as

$$\gamma(g^{\gamma(h)}h) = \gamma(h)\gamma(g)^{\gamma(h)} = \gamma(g)\gamma(h).$$

We have obtained

Theorem 5.2. *Let G be a finite group. The following data are equivalent.*

- (1) *A regular subgroup $N \trianglelefteq \text{Hol}(G)$, that is, an element of $\mathcal{K}(G)$.*

(2) A map $\gamma : G \rightarrow \text{Aut}(G)$ such that for $g, h \in G$ and $\beta \in \text{Aut}(G)$

$$(5.4) \quad \begin{cases} \gamma(gh) = \gamma(h)\gamma(g) \\ \gamma(g^\beta) = \gamma(g)^\beta. \end{cases}$$

Moreover, under these assumptions

(1) the assignment

$$g \circ h = g^{\gamma(h)}h.$$

for $g, h \in G$, defines a group structure (G, \circ) with the same unity as that of G .

(2) There is an isomorphism $\nu : (G, \circ) \rightarrow N$.

(3) For $g, h \in G$, one has

$$g^{\nu(h)} = g \circ h.$$

(4) Every automorphism of G is also an automorphism of (G, \circ) . In particular, ν is a homomorphism of groups with $\text{Aut}(G)$ as a group of operators.

In the following, when dealing with $N \in \mathcal{K}(G)$, we will be using the notation of Theorem 5.2 without further mention.

Proof. The last statement follows from

$$(g \circ h)^\beta = (g^{\gamma(h)}h)^\beta = (g^\beta)^{\gamma(h)^\beta}h^\beta = (g^\beta)^{\gamma(h^\beta)}h^\beta = g^\beta \circ h^\beta,$$

for $g, h \in G$ and $\beta \in \text{Aut}(G)$. \square

Let us exemplify the above for the case of the left regular representation. Consider the morphism

$$\begin{aligned} \iota : G &\rightarrow \text{Aut}(G) \\ y &\mapsto (x \mapsto y^{-1}xy), \end{aligned}$$

that is, $\iota(y) \in \text{Inn}(G)$ is conjugacy by y . If $N = \lambda(G)$, then we have for $y \in G$

$$\lambda(y) = \iota(y^{-1})\rho(y),$$

as for $z, y \in G$ we have

$$z^{\iota(y^{-1})\rho(y)} = yzy^{-1}y = yz = z^{\lambda(y)}.$$

Therefore $\gamma(y) = \iota(y^{-1})$, and

$$x \circ y = x^{\iota(y^{-1})}y = yx,$$

that is, (G, \circ) is the opposite group of G , according to

Definition 5.3. The *opposite* of the group G is the group obtained by exchanging the order of factors in the product of G .

Also, in [CC99] S. Carnahan and L. Childs prove that if G is a non-abelian finite simple group, then $\mathcal{H}(G) = \{\rho(G), \lambda(G)\}$. In our context, this can be proved as follows. If G is a non-abelian finite simple group, and $N \in \mathcal{H}(G)$, then the normal subgroup $\ker(\gamma)$ of G can only be either G or $\{1\}$. In the first case we have $x \circ y = x^{\gamma(y)}y = xy$ for $x, y \in G$, so that

$$x^{\nu(y)} = x \circ y = xy = x^{\rho(y)},$$

and $N = \rho(G)$. In the second case, γ is injective. Since we have

$$\gamma(x \circ y) = \gamma(x)\gamma(y) = \gamma(yx),$$

we obtain $x \circ y = yx$, so $N = \lambda(G)$ as we have just seen.

6. COMMUTATORS

In this section we assume we are in the situation of Theorem 5.2.

Let $\beta \in \text{Aut}(G)$, $g \in G$, and consider the commutator $[\beta, g^{-1}] = g^\beta g^{-1}$ taken in $\text{Aut}(G)G$. Using (5.4), we get

$$(6.1) \quad \gamma([\beta, g^{-1}]) = \gamma(g^\beta g^{-1}) = \gamma(g)^{-1} \gamma(g)^\beta = [\gamma(g), \beta].$$

In the particular case when $\beta = \iota(h)$, for some $h \in G$, we obtain

$$\gamma([h, g^{-1}]) = \gamma([\iota(h), g^{-1}]) = [\gamma(g), \iota(h)] = \iota([\gamma(g), h]),$$

that is

$$(6.2) \quad \gamma([h, g^{-1}]) = \iota([\gamma(g), h]).$$

From this identity we obtain

$$\begin{aligned} \iota([\gamma(g), h]) &= \gamma([h, g^{-1}]) = \gamma([g^{-1}, h])^{-1} = \\ &= \iota([\gamma(h^{-1}), g^{-1}])^{-1} = \iota([g^{-1}, \gamma(h^{-1})]), \end{aligned}$$

that is,

$$(6.3) \quad [\gamma(g), h] \equiv [g^{-1}, \gamma(h^{-1})] \pmod{Z(G)}$$

for all $g, h \in G$.

In the rest of the paper we will deal with the case of finite perfect groups, that is, those finite groups G such that $G' = G$. In this case, according to (6.2), we have $\gamma(G) \leq \text{Inn}(G)$.

7. PERFECT GROUPS

Let G be a non-trivial, finite, perfect group. We will determine $\mathcal{K}(G)$, and then discuss its relationship to $\mathcal{H}(G)$.

Recall that an automorphism β of a group G is said to be *central* if $[x, \beta] = x^{-1}x^\beta \in Z(G)$ for all $x \in G$. We will make use later of the following well known

Lemma 7.1. *The set of central automorphisms of a group G is the centralizer in $\text{Aut}(G)$ of $\text{Inn}(G)$.*

We record for later usage another couple of elementary, well-known facts.

Lemma 7.2. *Let G be a finite perfect group.*

- (1) $Z_2(G) = Z(G)$.
- (2) *A central automorphism of G is trivial.*

Proof. The first part is Grün's Lemma [Grü35].

For the second part, if β is a central automorphism of G , then

$$x \mapsto [x, \beta]$$

is a homomorphism from G to $Z(G)$. Since $G = G'$, this homomorphism maps G onto the identity. \square

We now show that an element $N \in \mathcal{K}(G)$ yields a direct product decomposition of $\text{Inn}(G)$.

Proposition 7.3. *Let G be a finite, perfect group, and $N \in \mathcal{K}(G)$.*

- (1) $Z(G) \leq \ker(\gamma)$.
- (2) $\text{Inn}(G) = \gamma(G) \times \iota(\ker(\gamma))$.

Later we will lift the direct product decomposition (2) of $\text{Inn}(G)$ to a central product decomposition of G (Theorem 7.5(1)).

Proof. For the first part, let $g \in Z(G)$. (6.3) yields $[\gamma(g), h] \in Z(G)$ for all $h \in G$, that is, $\gamma(g)$ is a central automorphism of G . By Lemma 7.2(2), $\gamma(g) = 1$.

For the second part, we first show that $\gamma(G)$ and $\iota(\ker(\gamma))$ commute elementwise. Let $g \in G$ and $k \in \ker(\gamma)$. The results of Section 6 yield

$$[\gamma(g), \iota(k)] = \iota([\gamma(g), k]) = \iota([g^{-1}, \gamma(k^{-1})]) = 1.$$

We now show that $\gamma(G) \cap \iota(\ker(\gamma)) = 1$. Write an element of the perfect group G as

$$x = \prod_{i=1}^n [h_i, g_i^{-1}],$$

for suitable $g_i, h_i \in G$.

Using the first identity of (5.4) we get first

$$(7.1) \quad \gamma(x) = \prod_{i=n}^1 \gamma([h_i, g_i^{-1}]) = \prod_{i=n}^1 [\gamma(g_i), \gamma(h_i^{-1})]$$

(note that the order of the product has been inverted by the application of γ).

Using (6.2) and the first identity of (5.4) we also get

$$\gamma(x) = \prod_{i=n}^1 \gamma([h_i, g_i^{-1}]) = \prod_{i=n}^1 \iota([\gamma(g_i), h_i]) = \iota\left(\prod_{i=n}^1 [\gamma(g_i), h_i]\right).$$

Now if $\gamma(x) \in \gamma(G) \cap \iota(\ker(\gamma))$, part (1) yields

$$\prod_{i=n}^1 [\gamma(g_i), h_i] \in \ker(\gamma).$$

We thus have, using (5.4) and (6.1)

$$1 = \gamma\left(\prod_{i=n}^1 [\gamma(g_i), h_i]\right) = \prod_{i=1}^n \gamma([\gamma(g_i), h_i]) = \prod_{i=1}^n [\gamma(h_i^{-1}), \gamma(g_i)] = \gamma(x)^{-1},$$

according to (7.1). Therefore $\gamma(x) = 1$, as claimed.

Finally we have, keeping in mind part (1),

$$|\gamma(G) \times \iota(\ker(\gamma))| = |\gamma(G)| \cdot \frac{|\ker(\gamma)|}{|Z(G)|} = |G/Z(G)| = |\text{Inn}(G)|,$$

so that $\gamma(G) \times \iota(\ker(\gamma)) = \text{Inn}(G)$. \square

Regarding $\text{Inn}(G)$ and G as groups with operator group $\text{Aut}(G)$, we note that the second equation of (5.4) implies that both $\gamma(G)$ and $\iota(\ker(\gamma))$ are $\text{Aut}(G)$ -invariant, and so are $H = \iota^{-1}(\gamma(G))$ and $\ker(\gamma)$. (Clearly the latter statement is the same as saying that H and $\ker(\gamma)$ are characteristic subgroups of G , but we prefer to use the same terminology of groups with $\text{Aut}(G)$ as a group of operators for both G and $\text{Inn}(G)$.)

We have $G = H \ker(\gamma)$. We claim that $[H, \ker(\gamma)] = 1$, that is, G is the central product of H and $\ker(\gamma)$, amalgamating $Z(G)$. We will need the following simple Lemma, which is hinted at by Joshua A. Grochow and Youming Qiao in [GQ13, Remark 7.6].

Lemma 7.4. *Let G be a group, and $H, K \leq G$ such that*

$$G/Z(G) = HZ(G)/Z(G) \times KZ(G)/Z(G).$$

Suppose $KZ(G)/Z(G)$ is perfect.

Then

- (1) K' is perfect, and
- (2) $[H, K] = 1$.

Proof. Since $KZ(G)/Z(G)$ is perfect, we have $KZ(G) = K'Z(G)$, so that $K' = K''$, and K' is perfect. As $[H, K'] = [H, K'Z(G)] = [H, KZ(G)] = [H, K] \leq Z(G)$, H induces by conjugation central automorphisms on K' , so that by Lemma 7.2(2) $[H, K] = [H, K'] = 1$. \square

In our situation, Lemma 7.4 implies that G is the central product of H and $\ker(\gamma)$, amalgamating $Z(G)$. Note that since $G = G' = H' \ker(\gamma)' = H' \ker(\gamma)$, we may replace H by H' , which is perfect (and might well be trivial).

We claim that $\gamma(y) = \iota(y^{-1})$ for $y \in H$.

Consider the isomorphism

$$\begin{aligned}\iota' : G/Z(G) &\rightarrow \text{Inn}(G) \\ yZ(G) &\mapsto \iota(y).\end{aligned}$$

For $y \in H$, write $\gamma(y) = \iota'((y^{-1}Z(G))^\sigma)$, for some map σ on $G/Z(G)$. The first equation of (5.4) implies that σ is a homomorphism. Moreover, σ is an automorphism of $G/Z(G)$, since if $y^{-1}Z(G) \in \ker(\sigma)$, then $\gamma(y) = 1$, so that $y \in H \cap \ker(\gamma) = Z(G)$, and $y^{-1}Z(G) = Z(G)$.

For $x \in H$ we have $\gamma(y^{\iota(x)}) = \gamma(y)^{\iota(x)}$, that is

$$\iota'((y^{-1}Z(G))^{\iota'(xZ(G))\sigma}) = \iota'((y^{-1}Z(G))^\sigma \iota'(xZ(G))).$$

Thus σ commutes with the inner automorphisms of the centerless group $HZ(G)/Z(G)$, so that by Lemma 7.1 σ is a central automorphism of the perfect group $HZ(G)/Z(G)$, and thus σ is trivial by Lemma 7.2(2). It follows that $\gamma(y) = \iota'(y^{-1}Z(G)) = \iota(y^{-1})$.

For $y \in H$ and $x \in \ker(\gamma)$ we have

$$x \circ y = x^{\gamma(y)}y = x^{\iota(y^{-1})}y = yxy^{-1}y = yx = y^{\gamma(x)}x = y \circ x.$$

Also, if $x, y \in H$ we have $x \circ y = x^{\gamma(y)}y = x^{y^{-1}}y = yx$.

In the following we will be writing the elements of G as pairs in $H \times \ker(\gamma)$, understanding that a pair represents an equivalence class with respect to the central product equivalence relation which identifies (xz, y) with (x, zy) , for $z \in Z(G)$.

We have obtained

Theorem 7.5. *Let G be a finite perfect group.*

- (1) *If $N \in \mathcal{K}(G)$, then G is a central product of its subgroups $H = \iota^{-1}(\gamma(G))'$ and $K = \ker(\gamma)$. Both H and K are $\text{Aut}(G)$ -subgroups of G .*
- (2) *For $x \in H$ we have $\gamma(x) = \iota(x^{-1})$.*
- (3) *(G, \circ) is also a central product of the same subgroups. If we represent the elements of G as (equivalence classes of) pairs in $H \times K$, then*

$$(7.2) \quad (x, y) \circ (a, b) = (ax, yb).$$

- (4) *For $(a, b) \in G$, the action of $\nu(a, b)$ on $(x, y) \in G$ is given by*

$$(x, y)^{\nu(a, b)} = (x, y) \circ (a, b) = (ax, yb),$$

that is, N induces the right regular representation on K , and the left regular representation on H .

We note the following analogue of Proposition 2.2(1) and (2).

Proposition 7.6. *Let G be a finite perfect group, and let $G = HK$ be a central decomposition, with $\text{Aut}(G)$ -invariant subgroups H, K . Consider the following two elements of $\mathcal{K}(G)$.*

- (1) N_1 , for which $\ker(\gamma_1) = KZ(G)$ and $H = \iota^{-1}(\gamma_1(G))'$, with $\gamma_1(x) = \iota(x^{-1})$ for $x \in H$, and associated group operation $(x, y) \circ_1 (a, b) = (ax, yb)$.
- (2) N_2 , for which $\ker(\gamma_2) = HZ(G)$ and $K = \iota^{-1}(\gamma_2(G))'$, with $\gamma_2(x) = \iota(x^{-1})$ for $x \in K$, and associated group operation $(x, y) \circ_2 (a, b) = (xa, by)$.

Then

- (1) $N_1^{\text{inv}} = N_2$.
- (2) $\text{inv} : (N_1, \circ_1) \rightarrow (N_2, \circ_2)$ is an isomorphism.
- (3) $C_{S(G)}(N_1) = N_2$ and $C_{S(G)}(N_2) = N_1$.

Proof. The proof is straightforward. If $N_i = \{\nu_i(a, b) : (a, b) \in G\}$ as in Section 4, we have

$$\begin{aligned} (x, y)^{\text{inv} \nu_1(a, b) \text{inv}} &= (x^{-1}, y^{-1})^{\nu_1(a, b) \text{inv}} = \\ &= (ax^{-1}, y^{-1}b)^{\text{inv}} = (xa^{-1}, b^{-1}y) = (x, y)^{\nu_2((a, b)^{\text{inv}})}, \end{aligned}$$

and then, as in Lemma 4.2, $\text{inv} : (N_1, \circ_1) \rightarrow (N_2, \circ_2)$ is an isomorphism. \square

We now give a description of all possible central product decompositions of the perfect group G as in Theorem 7.5.

We deal first with the particular case when $Z(G) = 1$, where we are able to show that $\mathcal{K}(G) = \mathcal{L}(G) = \mathcal{H}(G)$ and determine this set, and the group $T(G) = \text{NHol}(G)/\text{Hol}(G)$. When $Z(G)$ is allowed to be non-trivial, we are able to determine $\mathcal{K}(G)$. However, examples show that in this case $\mathcal{H}(G)$, $\mathcal{L}(G)$ and $\mathcal{K}(G)$ can be distinct, and we are unable at the moment to describe $T(G)$.

7.1. The centerless case. Suppose $Z(G) = 1$, so that $\iota : G \rightarrow \text{Inn}(G)$ is an isomorphism of $\text{Aut}(G)$ -groups.

Consider a Krull-Remak-Schmidt decomposition

$$G = A_1 \times A_2 \times \cdots \times A_n$$

of G as an $\text{Aut}(G)$ -group. Since $Z(G) = 1$, this is unique [Rob96, 3.3.8, p. 83]. Therefore the only way to decompose G as the ordered direct product of two characteristic subgroups H, K is by grouping together the A_i , so that there are 2^n ways of doing this. If $G = H \times K$ is one of these ordered decompositions, define an antihomomorphism $\gamma : G \rightarrow \text{Aut}(G)$ by $\gamma(k) = 1$ for $k \in K$, and $\gamma(h) = \iota(h^{-1})$, for $h \in H$. Then γ satisfies also the second identity of (5.4), and we have obtained an element $N \in \mathcal{K}(G)$ as in Theorem (7.5)(3). The involution $\vartheta \in \text{NHol}(G)$ given by $(h, k)^{\vartheta} = (h^{-1}, k)$, for $h \in H$ and $k \in K$ is an isomorphism $G \rightarrow (G, \circ)$. We have obtained

Theorem 7.7. *Let G be a finite perfect group with $Z(G) = 1$.*

- (1) *If $N \in \mathcal{K}(G)$, that is, $N \trianglelefteq \text{Hol}(G)$ is regular, then $N \in \mathcal{H}(G)$, that is, $N \cong G$.*

- (2) *If n is the length of a Krull-Remak-Schmidt decomposition of G as an $\text{Aut}(G)$ -group, then $\mathcal{H}(G)$ has 2^n elements.*
 (3) *$T(G)$ is an elementary abelian group of order 2^n .*

7.2. Non-trivial center. We now consider the situation when $Z(G)$ is (allowed to be) non-trivial.

We describe the elements N of $\mathcal{K}(G)$, in analogy with the centerless case.

As in the centerless case, we may consider the Krull-Remak-Schmidt decomposition

$$(7.3) \quad \text{Inn}(G) = A_1 \times A_2 \times \cdots \times A_n$$

of $\text{Inn}(G)$ as an $\text{Aut}(G)$ -group. This corresponds uniquely to the central product decomposition of G

$$(7.4) \quad G = B_1 B_2 \cdots B_n,$$

where $B_i = \iota^{-1}(A_i)'$ are *perfect* $\text{Aut}(G)$ -subgroups, which are centrally indecomposable as $\text{Aut}(G)$ -subgroups. Therefore the central product decomposition (7.4) is also unique. (Recall also that the Krull-Remak-Schmidt of G in terms of indecomposable $\text{Aut}(G)$ -subgroups is unique, because of [Rob96, 3.3.8, p. 83] and Lemma 7.2.(2).)

As in the centerless case, we obtain that every decomposition $G = H \ker(\gamma)$ as in Theorem 7.5 can be obtained by grouping together the B_i in two subgroups H and K , and then defining an antihomomorphism $\gamma : G \rightarrow \text{Aut}(G)$ by $\gamma(k) = 1$ for $k \in KZ(G)$, and $\gamma(h) = \iota(h^{-1})$, for $h \in H$, and then \circ as in (7.2). As in the centerless case, this yields an element $N \in \mathcal{K}(G)$. Moreover (G, \circ) is still a central product of H and $KZ(G)$, with \circ as in Theorem 7.5(3).

We have obtained the following weaker analogue of Theorem 7.7.

Theorem 7.8. *Let G be a finite perfect group.*

If n is the length of a Krull-Remak-Schmidt decomposition of $\text{Inn}(G)$ as an $\text{Aut}(G)$ -group, then $\mathcal{K}(G)$ has 2^n elements, that is, there are 2^n regular subgroups $N \trianglelefteq \text{Hol}(G)$.

Write $Y = H \cap KZ(G)$. As the subgroups H and $KZ(G)$ are characteristic in G , we obtain that the elements of $\text{Aut}(G)$ can be described via the set of pairs

$$\{(\sigma, \tau) : \sigma \in \text{Aut}(H), \tau \in \text{Aut}(KZ(G)), \sigma|_Y = \tau|_Y\}.$$

Applying the element of $\text{Aut}(G)$ described by the pair (σ, τ) to (7.2), we obtain

$$\begin{aligned} ((x, y) \circ (a, b))^{(\sigma, \tau)} &= (ax, yb)^{(\sigma, \tau)} = ((ax)^\sigma, (yb)^\tau) = \\ &= (a^\sigma x^\sigma, y^\tau b^\tau) = (x^\sigma, y^\tau) \circ (a^\sigma, b^\tau) = (x, y)^{(\sigma, \tau)} \circ (a, b)^{(\sigma, \tau)}. \end{aligned}$$

This shows that $\text{Aut}(G) \leq \text{Aut}(G, \circ)$.

However, the latter group might well be bigger than the former. This is shown by the following

Proposition 7.9. *There exists perfect, centrally indecomposable groups Q_1, Q_2, Q_3 , and a central product $G = Q_1Q_2Q_3$, such that*

- (1) *each Q_i is characteristic in G ,*
- (2) *in the group (G, \circ) obtained by replacing Q_1 with its opposite, the subgroups Q_1 and Q_2 are exchanged by an automorphism of (G, \circ) , and thus are not characteristic in (G, \circ) .*

Clearly the automorphism of the second condition lies in $\text{Aut}(G, \circ) \setminus \text{Aut}(G)$. This example shows that $\mathcal{L}(G)$ may well be a proper subset of $\mathcal{K}(G)$.

Clearly $N \in \mathcal{L}(G)$ if and only if $\text{Aut}(G) = \text{Aut}(G, \circ)$. However, in this general situation $\mathcal{H}(G)$ might be a proper subset of $\mathcal{L}(G)$. This is shown by the following

Proposition 7.10. *There exist perfect, centrally indecomposable groups Q_1, Q_2 , and a central product $G = Q_1Q_2$ such that*

- (1) *Q_1, Q_2 are characteristic in G , and in the group (G, \circ) obtained from G by replacing Q_1 with its opposite;*
- (2) *G is not isomorphic to (G, \circ) .*

Thus if N is the regular subgroup corresponding to (G, \circ) of this Proposition, we have $N \in \mathcal{L}(G) \setminus \mathcal{H}(G)$.

So on the one hand not all central products decompositions lead to regular subgroups N which are isomorphic to G . And even when N is isomorphic to G , it is not clear whether there is an involutory isomorphism, and therefore it is not clear to us at the moment whether $T(G)$ is elementary abelian in this general case.

To construct the groups of Proposition 7.9 and 7.10, we rely on the following family of examples, based on a construction that we have learned from Derek Holt.

Proposition 7.11. *There exists a family of pairwise non-isomorphic, perfect, centrally indecomposable groups L_p , for $p \equiv 1 \pmod{3}$ a prime, with the following properties:*

- (1) *$Z(L_p)$ is of order 3, and*
- (2) *$\text{Aut}(L_p)$ acts trivially on $Z(L_p)$.*

This is proved in Section 8.

Proof of Proposition 7.9. Let Q_1, Q_2 be two copies of one of the groups of Proposition 7.11, and Q_3 another group as in Proposition 7.11, not isomorphic to Q_1, Q_2 .

Fix an isomorphism $\zeta : Q_1 \rightarrow Q_2$. If $Z(Q_1) = \langle a_1 \rangle$, let $a_2 = a_1^\zeta$. Let $Z(Q_3) = \langle b \rangle$.

Consider the central product $G = Q_1Q_2Q_3$, amalgamating $a_2 = a_1^{-1} = b$.

Q_3 is clearly characteristic in G . If there is an automorphism α of G that exchanges Q_1 and Q_2 , then $\alpha|_{Q_1} \zeta^{-1}$ is an automorphism of Q_1 , and thus maps a_1 to a_1 . Therefore α maps a_1 to $a_1^\zeta = a_2 = a_1^{-1}$. But $\alpha|_{Q_3}$ is an automorphism of Q_3 , and thus fixes $b = a_1$, a contradiction.

Consider now the group (G, \circ) obtained by replacing Q_1 with its opposite. Now the map which is the identity on Q_3 , $\zeta \text{ inv}$ on Q_1 and $\zeta^{-1} \text{ inv}$ on Q_2 induces an automorphism of (G, \circ) which exchanges Q_1 and Q_2 . In fact we have for $x, y \in Q_1$

$$(x \circ y)^{\zeta \text{ inv}} = (yx)^{\zeta \text{ inv}} = (y^\zeta x^\zeta)^{\text{inv}} = x^{\zeta \text{ inv}} y^{\zeta \text{ inv}},$$

and for $x, y \in Q_2$

$$\begin{aligned} (x \circ y)^{\zeta^{-1} \text{ inv}} &= (xy)^{\zeta^{-1} \text{ inv}} = \\ &= (x^{\zeta^{-1}} y^{\zeta^{-1}})^{\text{inv}} = y^{\zeta^{-1} \text{ inv}} x^{\zeta^{-1} \text{ inv}} = x^{\zeta^{-1} \text{ inv}} \circ y^{\zeta^{-1} \text{ inv}}. \end{aligned}$$

Moreover $a_1^{\zeta \text{ inv}} = a_2^{\text{inv}} = a_2^{-1} = a_1$, and $a_2^{\zeta^{-1} \text{ inv}} = a_1^{\text{inv}} = a_1^{-1} = a_2$, which is compatible with the identity on Q_3 . \square

Proof of Proposition 7.10. Let Q_1, Q_2 to be two non-isomorphic groups as in Proposition 7.11, and let $G = Q_1 Q_2$, amalgamating the centers. Consider the group (G, \circ) obtained by replacing Q_1 with its opposite. If the map $\vartheta : G \rightarrow G$ is an isomorphism of G onto (G, \circ) , clearly it has to map each Q_i to itself. Then ϑ induces an anti-automorphism on Q_1 , thus inverting $Z(G)$, and an automorphism on Q_2 , thus fixing $Z(G)$ elementwise, a contradiction. \square

8. PROOF OF PROPOSITION 7.11

Consider the groups $T_p = \text{PSL}(3, p)$, where $p \equiv 1 \pmod{3}$ is a prime, and let \mathbb{F}_p the field with p elements.

It is well known ([Wil09, Theorem 3.2]) that the outer automorphism group of T_p is isomorphic to S_3 , where an automorphisms Δ of order 3 is diagonal, obtained via conjugation with a suitable $\delta \in \text{PGL}(3, p)$, and one of the involutions is the transpose inverse automorphisms \top .

Moreover, the Schur multiplier of T_p has order 3 [Wil09, 3.3.6], it is inverted by \top , and clearly centralized by Δ .

Let P be the natural \mathbb{F}_p -permutation module of T_p in its action on the points of the projective plane. P is the direct sum of a copy of the trivial module, and of a module N . The structure of N is investigated in [ZS90], [Abd97]. In particular, it is shown in [Abd97] that N has a unique composition series $\{0\} \subset N_1 \subset N$, such that N_1 and $N_2 = N/N_1$ are dual to each other, exchanged by \top , but not isomorphic to each other. Note that Δ still acts on P , as it comes from conjugation with an element $\delta \in \text{PGL}(3, p)$, and thus also acts on N_1 and N_2 .

Consider the natural semidirect product S_p of N_1 by T_p . It has been shown by K.I. Tahara [Tah72] that the Schur multiplier of T_p is a direct summand of the Schur multiplier of S_p , that is, $M(S_p) = M(T_p) \oplus K$

for some K . We may thus consider the central extension L_p of $M(T_p)$ by S_p , which is the quotient of the covering group of S_p by K . Thus $Z(L_p) = M(T_p)$ (Derek Holt has shown to us calculations for small primes, based on the description of [Tah72], which appear to indicate that actually $K = \{0\}$ here.)

An automorphism σ of L_p induces automorphisms α of N_1 and β of T_p . Abusing notation slightly, we have, for $n \in N_1$ and $h \in T_p$,

$$(n^h)^\alpha = (n^h)^\sigma = (n^\sigma)^{h^\sigma} = (n^\alpha)^{h^\beta}.$$

If β is an involution in $\text{Out}(T_p)$, say $\beta = \Delta^{-1}\top\Delta$, then we have

$$h^\beta = \delta^{-2}h^\top\delta^2,$$

so that

$$(n^h)^{\alpha\Delta^{-2}} = (n^{\alpha\Delta^{-2}})^{h^\top},$$

that is, $\alpha\Delta^{-2}$ is an isomorphism of N_1 with its dual, a contradiction. Then $\text{Aut}(L_p)$ induces on T_p only inner automorphisms and at most outer automorphisms of order 3, all of which centralize $M(T_p) = Z(L_p)$.

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